UNIVERSITY OF LONDON

ST2133 ZB

BSc DEGREES AND GRADUATE DIPLOMAS IN ECONOMICS, MANAGEMENT, FINANCE AND THE SOCIAL SCIENCES, THE DIPLOMA IN ECONOMICS AND SOCIAL SCIENCES AND THE CERTIFICATE IN EDUCATION IN SOCIAL SCIENCES

Summer 2021 Online Assessment Instructions

ST2133 Advanced statistics: distribution theory

Friday, 28 May 2021: 06:00 – 10:00 (BST)

The assessment will be an **open-book take-home online assessment within a 4-hour window**. The requirements for this assessment remain the same as the closed-book exam, with an expected time/effort of **2 hours.**

Candidates should answer all **FOUR** questions: **QUESTION 1** of Section A (40 marks) **and all THREE** questions from Section B (60 marks in total). **Candidates are strongly advised to divide their time accordingly.**

You should complete this paper **using pen and paper.** Please use **BLACK INK** only.

Handwritten work then needs to be scanned, converted to PDF and then uploaded to the VLE as **ONE individual file** including the coversheet. Each scanned sheet should have your **candidate number** written clearly in the header. Please **do not write your name anywhere** on your submission.

You have until 10:00 (BST) on Friday, 28 May 2021 to upload your file into the VLE submission portal. However, you are advised not to leave your submission to the last minute.

Workings should be submitted for all questions requiring calculations. Any necessary assumptions introduced in answering a question are to be stated.

You may use *any* calculator for any appropriate calculations, but you may not use any computer software to obtain solutions. Credit will only be given if all workings are shown.

If you think there is any information missing or any error in any question, then you should indicate this but proceed to answer the question stating any assumptions you have made.

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The assessment has been designed with a duration of 4 hours to provide a more flexible window in which to complete the assessment and to appropriately test the course learning outcomes. As an open-book exam, the expected amount of effort required to complete all questions and upload your answers during this window is no more than 2 hours. Organise your time well.

You are assured that there will be no benefit in you going beyond the expected 2 hours of effort. Your assessment has been carefully designed to help you show what you have learned in the hours allocated.

This is an open book assessment and as such you may have access to additional materials including but not limited to subject guides and any recommended reading. But the work you submit is expected to be 100% your own. Therefore, unless instructed otherwise, you must not collaborate or confer with anyone during the assessment. The University of London will carry out checks to ensure the academic integrity of your work. Many students that break the University of London's assessment regulations did not intend to cheat but did not properly understand the University of London's regulations on referencing and plagiarism. The University of London considers all forms of plagiarism, whether deliberate or otherwise, a very serious matter and can apply severe penalties that might impact on your award.

The University of London 2020-21 Procedure for the consideration of Allegations of Assessment Offences is available online at:

Assessment Offence Procedures - University of London

Section A

Answer all $\mathbf{three}$ parts of question 1 (40 marks in total)

1. (a) The probability mass function of a random variable X is given by

$$p_X(x) = \frac{c\mu^x e^{-\mu}}{x!}, \quad x = 1, 2, \cdots ,$$

where $\mu > 0$ and c is a constant.

i. Find the value of c. [4 marks]

ii. Find $P(X \geq 2.5)$. You can leave your answer in terms of c. [3 marks]

iii. Find $E(X)$ and $E(X(X-1))$. Hence find Var(X). You can leave your answer in terms of c. [7 marks]

(b) Let X follow a standard normal distribution. Define $W = X^2$.

i. Show that the moment generating function of W is given by

$$M_W(t) = \frac{1}{(1-2t)^{1/2}}, \quad t < 1/2.$$

[5 marks]

ii. State the Markov inequality for a non-negative random variable Y. [1 mark]

iii. Show that, for $0 < t < 1/2$ and $a > 0$,

$$P(W \geq a) \leq \frac{e^{-ta}}{(1-2t)^{1/2}}.$$

[6 marks]

(c) In a game, a fair die is thrown independently n times. Let X be the total number of throws showing 3 or higher. If $X \leq 1$, you lose the game.

i. Show that

$$P(\text{losing a game}) = \frac{2n+1}{3^n}.$$

[4 marks]

ii. Suppose m independent games are played. Write down the probability mass function of Y, where Y denotes the number of games lost. [2 marks]

iii. Now suppose that the number of games played, M, follows a Poisson distribution with mean μ (so the answer to part ii. is the conditional mass function of Y given $M = m$). Show that

$$p_Y(y) = \frac{(\mu p)^y e^{-\mu p}}{y!}, \text{ where } p = \frac{2n+1}{3^n}.$$

[8 marks]

Section B

Answer all three questions in this section (60 marks in total)

2. The joint probability density for the random variables X and Y is given by

$$f_{X,Y}(x,y) = \frac{c}{x(2\log x + \log y)}, \quad 1 < x < y < 4.$$

Define the transformation

$$U = X,\ V = \frac{\log X}{\log Y}.$$

(a) Show that the joint density $f_{U,V}(u,v)$ of U and V is given by

$$f_{U,V}(u,v) = \frac{cu^{1/v-1}}{v(1+2v)}, \quad 1 < u < 4^v,\ 0 < v < 1.$$

You should show clearly how you arrive at the region on the (u,v) plane where the density is defined. [8 marks]

(b) Work out the marginal density of V in terms of c. [2 marks]

(c) Show that the conditional density of U given $V = v$ is given by

$$f_{U|V}(u|v) = \frac{1}{3v}u^{1/v-1},\ 1 < u < 4^v.$$

[2 marks]

(d) Find $E(VU^{-1/V}|V=v)$. Hence, given $E(V) = 1/\log 3 - 1/2$, find

$$E\left(\frac{\log X}{Y\log Y}\right).$$

[8 marks]

3. Let $X_1, X_2, \ldots$, be a sequence of independent and identically distributed random variables. Let N be Poisson distributed with mean μ and is independent of the X_i's. Define

$$W = \sum_{i=1}^{N} X_i.$$

We define $W = 0$ if $N = 0$.

(a) Suppose each X_i is normally distributed with mean 0 and variance 1. Work out the moment generating function for W given N. [4 marks]

(b) Show that the moment generating function of W is given by

$$M_W(t) = \exp(\mu e^{t^2/2} - \mu),\ t \in \mathbb{R}.$$

[5 marks]

(c) Calculate the mean and variance of W. [5 marks]

(d) Now consider $Z = NX_1$. Find the mean and variance of Z. [6 marks]

4. A county is made up of three (mutually exclusive) communities A, B and C, with proportions of them given by the following table:

Community	A	B	C
Proportion	0.1	0.4	0.5

Given a person belonging to a certain community, the chance of that person being vaccinated is given by the following table:

Community given	A	B	C
Chance of being vaccinated	0.9	0.8	0.7

(a) We choose a person from the county at random. What is the probability the person is vaccinated?

[5 marks]

(b) We choose a person from the county at random. Find the probability that the person is in community B given that we know the person is not vaccinated.

[4 marks]

(c) If a person is vaccinated, the probability that they eventually show symptoms is 0.1, while the probability is 0.9 for a non-vaccinated person. For a person who eventually shows symptoms, the waiting time, T, until the symptoms appear is exponentially distributed, with rate 1/5 if they are vaccinated (i.e. $T \sim$ Exp(1/5)), and rate 1 if they are not.

i. Given that a person eventually shows symptoms, but these symptoms have not yet appeared at time $T = 1$, find the probability that this person is vaccinated. [6 marks]

ii. Given symptoms are shown eventually, find the mean of T. [5 marks]

END OF PAPER